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ON SHORT EDGES IN STRAIGHT-EDGE TRIANGULATIONS

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Abstract.

If a triangulation is drawn with straight edges, the ratio of the lengths of the shortest and the longest edges does not have to go to zero, even if the number of vertices goes to infinity. In this paper bounds are given for the above ratio, when certain restrictions are placed on the maximum degree of the triangulation.

A triangulation of the plane is a planar graph in which each region, including the infinite one, is bounded by exactly three edges. For standard terminology in graph theory, see, for example, [1]. The triangulation is said to be straight-edged if the graph is drawn with line segments as edges, so that it represents a triangle which is subdivided into smaller triangles. In this paper, a triangulation is always taken to mean a straight-edged triangulation.

Denote by T_n a triangulation with n-vertices and let $d(T_n)$ and $l(T_n)$ respectively by the lengths of the longest and shortest edges of T_n. The ratio $l(T_n)/d(T_n)$ is denoted by $\alpha(T_n)$.

As the number of vertices in a triangulation increases, the area of the smallest triangle divided by the area of the outer triangle must decrease to zero. However, such is not the case with the length of the shortest edge.

We constructed triangulations T_n, with n arbitrarily large satisfying $\alpha(T_n) > 1/4$. Two persons at a seminar of Micha A. Perles in Jerusalem ([2]), after seeing our construction, provided a better one. Unfortunately, their names are not known.

Figure 1 is an example of such a triangulation. The large triangle has vertices C at $(0,0)$, D at $(3,0)$ and A_1 on $x = 1$. B_1 is any point on $x = 2$ within A_1CD and it is joined to A_1, C and D. A_2 is any point on $x = 1$ within B_1CD, and it is joined to B_1, C and D. The point B_2 is chosen in a similar manner. The construction can be continued indefinitely. The length $l(T_n)$ is greater than 1 regardless of how large

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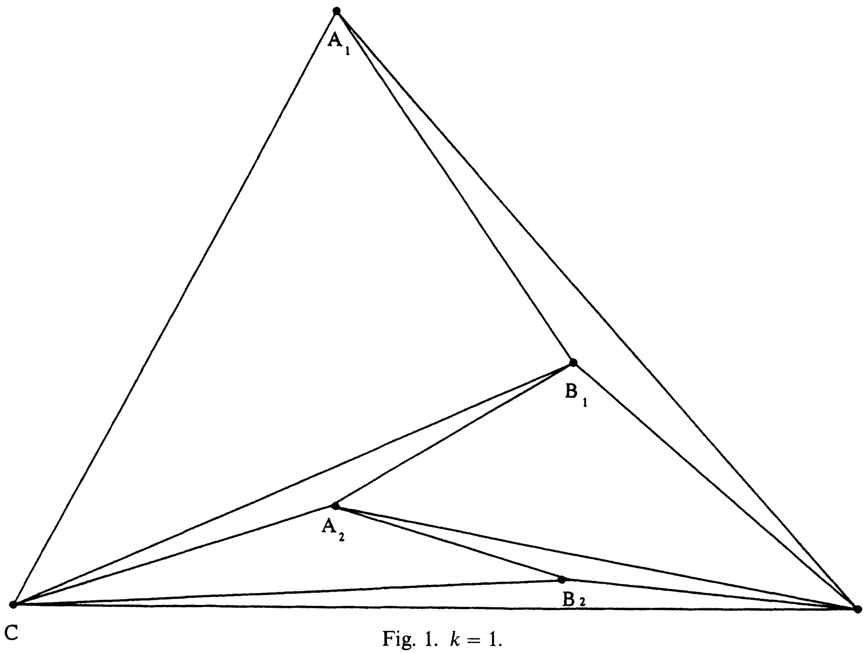

Fig. 1. $k = 1$.

n becomes. By compressing the large triangle towards CD if necessary, CD will be the longest edge. Hence, $d(T_n) = 3$ and $\alpha(T_n) > 1/3$.

Note that the above bound is the sharpest possible. Here is a sketch of the proof:

Assume that $d(T_n) = 1$, so that the area of the large triangle is at most $1/2$. It is divided into exactly $2n$-5 smaller triangles. It follows that at most $c\sqrt{n}$ of them can have an area exceeding $1/\sqrt{n}$, where c is a positive constant. All others, that is at least $2n$-5-$c\sqrt{n}$, have one angle very close to π and two angles very close to 0 when n is large and $\alpha(T_n)$ remains above a constant value (since the area of a triangle with sides a and b and angle θ between them is $(ab\sin\theta)/2$).

Among these triangles, one can find two with a common edge, say xvy and yvz, such that angle xvy is very close to π and angle yvz is very close to 0. Either x-v-y-z or x-v-z-y is almost a straight line. Since $d(T_n) = 1$, we cannot have $l(T_n) > 1/3 + \varepsilon$ for any fixed positive ε and large n.

In Figure 1, both C and D are joined to all other vertices as well as to each other. It is natural to ask what bounds can be obtained for $\alpha(T_n)$, when certain restrictions are placed on $\Delta(T_n)$, the maximum degree of T_n.

THEOREM A. *Let k be a fixed positive integer. Then, for an infinite number of values of the integer n, there exists a triangulation T_n which satisfies*

$$\alpha(T_n) > 1/(2k+1), \textit{ with } \Delta(T_n) < 2(n/2)^{1/k} + 2.$$

PROOF. Let $n = 1 + 2(1 + (m-1) + \ldots + (m-1)^k)$, where m is a positive integer; note that $2(m-1)^k < n$. A triangulation T_n with the desired properties is constructed.

The argument is inductive. The case $k = 1$ has already been dealt with in Figure 1. To give a clearer illustration of the general construction, consider the case $k = 2$. Figure 2 illustrates the case $m = 2$.

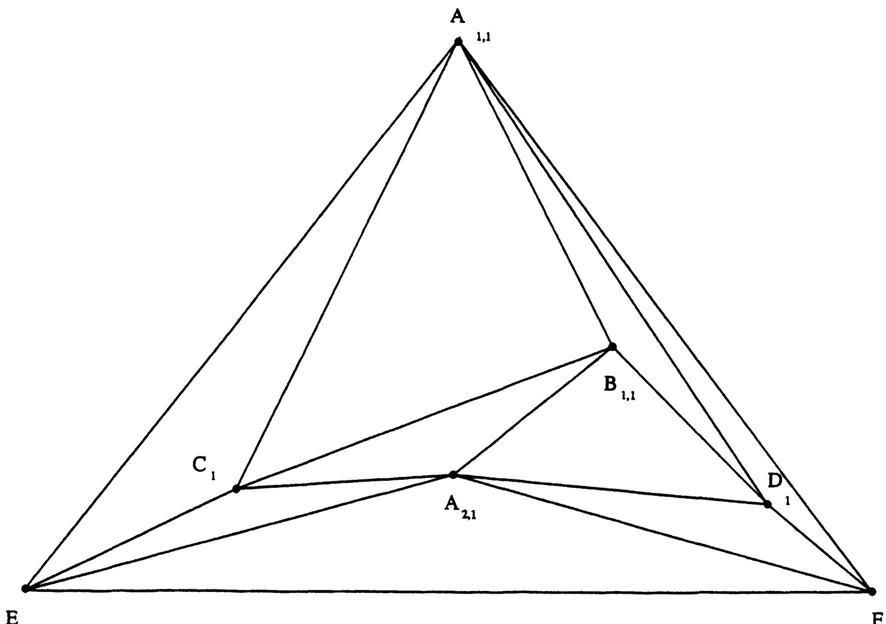

Fig. 2. $k = 2$.

Here, the large triangle has vertices E at $(0, 0)$, F at $(5, 0)$ and $A_{1,1}$ on $x = 2$. Take C_1 on $x = 1$ and D_1 on $x = 4$ such that C_1D_1 is parallel to EF and lies within $A_{1,1}EF$. Join C_1 to E and $A_{1,1}$, and join D_1 to F and $A_{1,1}$. We now triangulate $A_{1,1}C_1D_1$ as in the case $k = 1$, using the points $A_{1,2}A_{1,3}, \ldots, A_{1,m-1}$ on $x = 2$ and the points $B_{1,1}, B_{1,2}, \ldots, B_{1,m-1}$ on $x = 3$. Note, however, that C_1 is not joined to D_1.

Take $A_{2,1}$ on $x = 2$ within $B_{1,m-1}C_1D_1$ and join it to all three vertices as well as E and F. Take C_2 on $x = 1$ and D_2 on the line $x = 4$ such that C_2D_2 is parallel to EF and lies within $A_{2,1}EF$. The construction is repeated until $A_{m,1}$ is chosen and joined to E and F.

The total number of vertices is exactly equal to n. The degree of $A_{1,1}$ (and of $A_{m,1}$) is 5. The degree of $A_{i,1}$ is 8 for $2 \leqq i \leqq m-1$. The degree of $A_{i,j}$ (and of $B_{i,j}$)

is 4 for all other values of i and j. The degree of C_i (and of D_i) is $2m$ for all i, as is the degree of E (and of F). Hence, $\Delta(T_n) = 2m < 2\sqrt{n/2} + 2$. By compressing the large triangle towards EF if necessary, we can make EF the longest edge. Hence, $d(T_n) = 5$. Since $l(T_n) > 1$, $\alpha(T_n) > 1/5$.

We now consider the general case. Let the large triangle be $A_{1,1}EF$, with E at $(0,0)$, F at $(2k+1, 0)$ and $A_{1,1}$ on $x = k$. Take C_1 on $x = 1$ and D_1 on $x = 2k$ such that C_1D_1 is parallel to EF and lies within $A_{1,1}EF$. Join C_1 to E and $A_{1,1}$, and join D_1 to F and $A_{1,1}$. We can triangulate $A_{1,1}C_1D_1$ as in the case $k-1$. We now take an appropriate point $A_{2,1}$ on $x = k$ and continue with the inductive construction.

The total number of vertices in this triangulation is exactly n. It is routine to verify that, as in the case $k = 2$, none of the vertices has a degree exceeding $2m < 2(n/2)^{1/k} + 2$ and that $\alpha(T_n) > 1/(2k+1)$. This completes the proof of Theorem A.

THEOREM B. *For fixed positive integer k and n sufficiently large, every T_n satisfies $\alpha(T_n) < 1/k$, provided that $\Delta = \Delta(T_n) < (\pi n/6k^2)^{1/(k+1)}$.*

PROOF. The proof uses an indirect argument. Consider any T_n with $\Delta(T_n)$ as provided. Assume that $d(T_n) = 1$, so that $l(T_n) = \alpha(T_n)$. Suppose that $\alpha(T_n) \geqq 1/k$, and derive a contradiction.

We call a triangle in T_n good if its area is less than $S/2$, where $S = 5\Delta^k/n$. Note that S is very close to 0 when n is sufficiently large.

Call an angle θ thin if it satisfies $0 < \theta < 6/5\, k^2S$ and call it thick if $\pi - 6/5\, k^2S < \theta < \pi$.

Let a and b be the lengths of two edges of a good triangle, and let θ be the angle between them. Then $ab \sin\theta < S$. Since we assume that $a \geqq 1/k$ and $b \geqq 1/k$, $\sin\theta < k^2S$. Since k is fixed, $\sin\theta$ is very close to 0 and it follows that θ is either thin or thick. Thus each good triangle has exactly one thick and two thin angles.

Consider a vertex v surrounded by good triangles. Then every angle formed by two consecutive edges at v in clockwise order is either thick or thin, with at most two thick angles formed by consecutive edges at v. The sum of all the thin angles formed by consecutive edge at v is less than $\Delta(6/5)k^2S < \pi$. Hence, there are exactly two thick angles formed by consecutive edges at v. It follows that for any edge incident with v, there is another edge incident with v and forming a thick angle with the given edge.

Note that the two edges forming a thick angle are not required to lie on the same triangle of T_n.

Let V denote the set of vertices not surrounded by good triangles. In particular, the three vertices of the large triangle are in V. Since the area of this triangle is at most $1/2$, there are at most $1/S$ triangles which are not good, so that $|V| \leqq 3(1 + 1/S)$. The total number of vertices accessible from V by a path of length not exceeding k edges is at most $3(1 + 1/S)(1 + \Delta^2 + \ldots + \Delta^k) < 5\Delta^k/S = n$, since $\Delta \geqq 3$ and S is small.

This means that there exists a vertex x_0 which is inaccessible from V by any path of length not exceeding k edges. In particular, x_0 is surrounded by good triangles. Let x_1 be any vertex adjacent to x_0. Then x_1 is also surrounded by good triangles. Hence, among vertices adjacent to x_1, we can choose x_2 such that angle $x_0x_1x_2$ is thick. We can continue to choose vertices in this manner until we have x_{k+1}, which may no longer be surrounded by good triangles.

By our assumption x_ix_{i+1} has length at least $1/k$, for $0 \leqq i \leqq k$. Thus, the distance between x_0 and x_{k+1} is at least $(k + 1 - \frac{1}{2})/k$. It follows that the distance between x_0 and x_{k+1} exceeds 1, contradicting $d(T_n) = 1$. This completes the proof of Theorem B.

Our results are open for improvements. For example, Theorem A shows by example that $\alpha(T_n) > \frac{1}{5}$ and $\Delta(T_n) < 1.42n^{1/2}$ is possible, whereas Theorem B shows that $\alpha(T_n) > \frac{1}{5}$ and $\Delta(T_n) < 0.52n^{1/6}$ is impossible. Thus there are gaps in the size of $\Delta(T_n)$ for which the situation is still unresolved.

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REFERENCES

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2. M. Perles, *Private communication.*

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